ECC optimization on Sandy Bridge

The cost of cofactor $h = 1$

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Outline

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$$\mathcal{E} : y^2 = x^3 + ax + b$$

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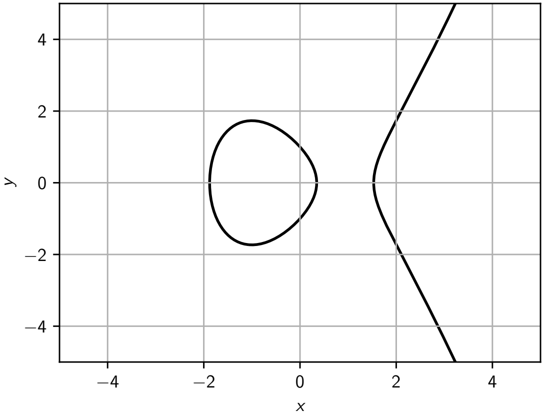

Elliptic curves: addition

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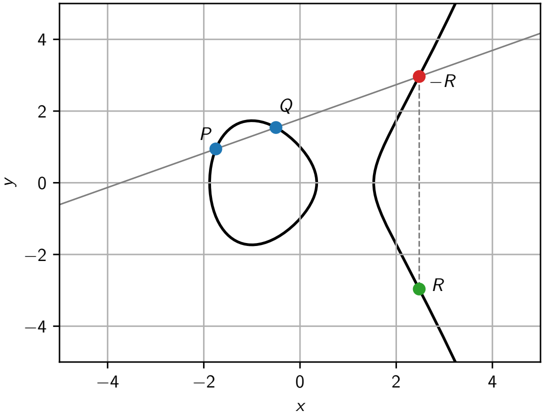

Elliptic curves: doubling

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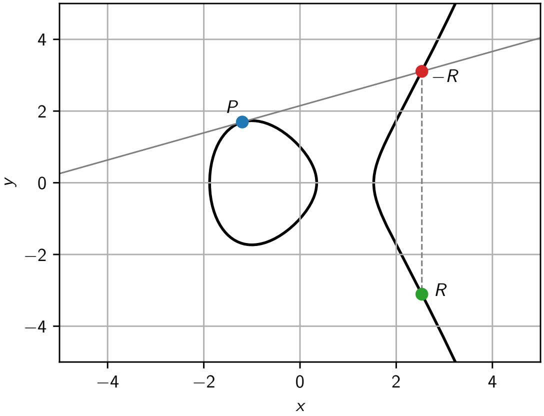

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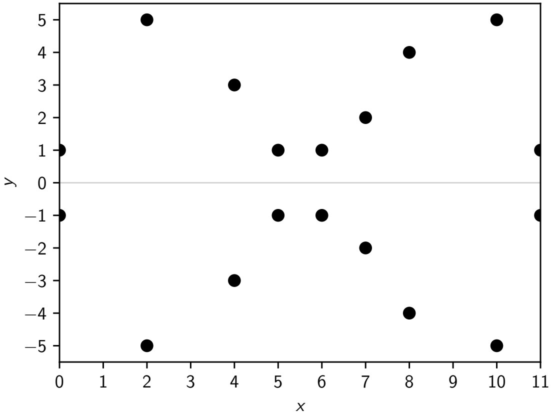
  - Define $P + \mathcal{O} = P$
- Curve equation: $\mathcal{E} : y^2 = x^3 + ax + b$
- Coordinates are defined over a field $\mathbb{F}_q$
  - I.e. integers modulo q

Elliptic curves: actually

$$\mathcal{E} : y^2 = x^3 - 3x + 1 \text{ defined over } \mathbb{F}_{11}$$



Elliptic curves: actual addition

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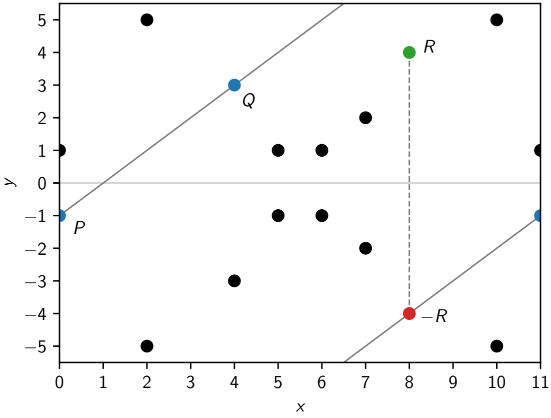

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- We can do arithmetic with these rules! :)

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- Discrete log problem:
  given P, Q where $[k]P = Q$, hard to find k

Elliptic curves are cyclic

- Points form a cycle: $\mathcal{O} \xrightarrow{+P} P \xrightarrow{+P} [2]P \xrightarrow{+P} [3]P \xrightarrow{+P} \ldots \xrightarrow{+P} [n-1]P \xrightarrow{+P} \mathcal{O}$

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- The **order** n should contain a large prime factor
- Only *one* cycle if n is prime

Cofactors

- If n is **not** a prime
  Then $n = h \cdot \ell$
- I.e. small loops are possible:
  E.g. if $4|n$, then there is a point T_4: $\underbrace{\mathcal{O} \xrightarrow{+T_4} T_4 \xrightarrow{+T_4} [2]\,T_4 \xrightarrow{+T_4} [3]\,T_4 \xrightarrow{+T_4} \mathcal{O}}_{\text{only 4 steps!}}$

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  - I.e. sometimes it's the opposite of harmless

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- 2014: Monero cryptocurrency
  - Uses Curve25519
- 2017: vulnerability in Monero found
  - Allowed anyone to create coins out of thin air

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Monero transactions

- Have generators G_1, G_2; private key x; public key P; key image I.
- $\text{SIGN}_x(m)$
  - Sign m with private key x
  - Choose commitment $u \in_R h\mathbb{Z}_\ell$
  - Compute $a_2 = [u]G_2$; $c = H(m, a_1, a_2)$; $r = u + cx$
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- $\text{VERIFY}_{P,I}(m, s)$
  - $[r]G_1 \stackrel{?}{=} a_1 + [c]P$
  - $[r]G_2 \stackrel{?}{=} a_2 + [c]I$
  - I unique?

Attacking Monero signatures

- **Challenge.** Find some signature+keypair a_2, c, r, and I, s.t.

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$$a_2 + [c]I' = a_2 + [c](I + T_\alpha)$$

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Easy fix:

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  - Fun fact: this check makes the verification $2\times$ slower

Why didn't they validate points?

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Look at the docs:

How do I validate Curve25519 public keys?

Don't. The Curve25519 function was carefully designed to allow all 32-byte strings as Diffie-Hellman public keys. Relevant lower-level facts: the number of points of this elliptic curve over the base field is 8 times the prime 2^252 + 27742317777372353535851937790883648493; the number of points of the twist is 4 times the prime 2^253 -

(highlight added by me)

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- Better fix: **use a prime order curve**

Outline

Goal of this thesis

What is the actual performance benefit of Curve25519 over traditional (Weierstrass) curves?

Our contribution

Our research:

- Implement variable base-point scalar multiplication
  - That is the algorithm for computing $[k]P$,
  - for a prime-order curve,
  - that looks similar to Curve25519,
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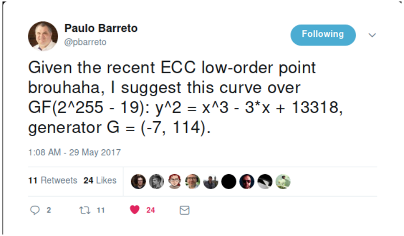
- Compare performance with Curve25519 (Sandy2x)

Selecting a curve



- I.e. $\mathcal{E} : y^2 = x^3 - 3x + 13318$, defined over $\mathbb{F}_{2^{255}-19}$.

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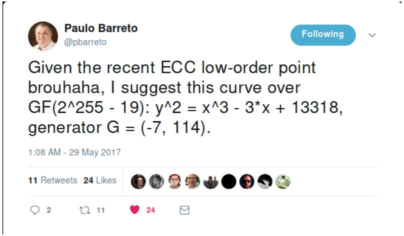
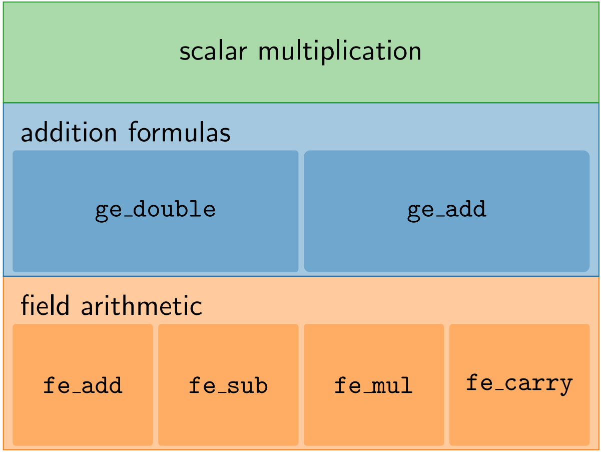


- I.e. $\mathcal{E} : y^2 = x^3 - 3x + 13318$, defined over $\mathbb{F}_{2^{255}-19}$.
- Prime order curve; same field as Curve25519

Scalar multiplication overview



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  - I.e. $f = f_0 + f_1 + ... + f_{11}$

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  - $(f+g)_i = f_i + g_i$
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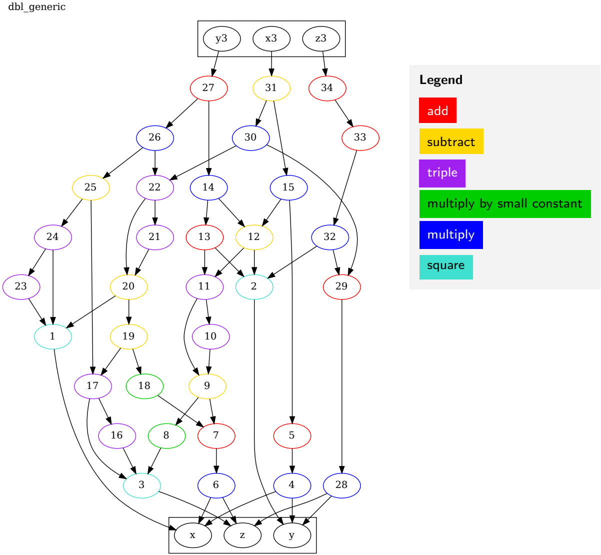
- Multiplication
  - $(f \cdot g)_k = \sum_{i+j=k} f_i g_i + \sum_{i+j=k+12} \left(2^{-255} \cdot 19\right) f_i g_i$
  - Optimized using Karatsuba's multiplication

Addition formulas

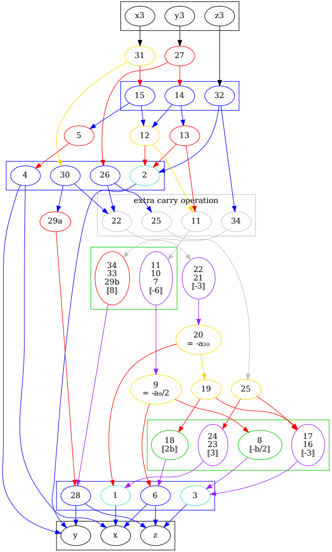
- Use Renes-Costello-Batina formulas
- Rewrite using graphs into vectorized operations
- Implement using field arithmetic functions

Point doubling



Point doubling

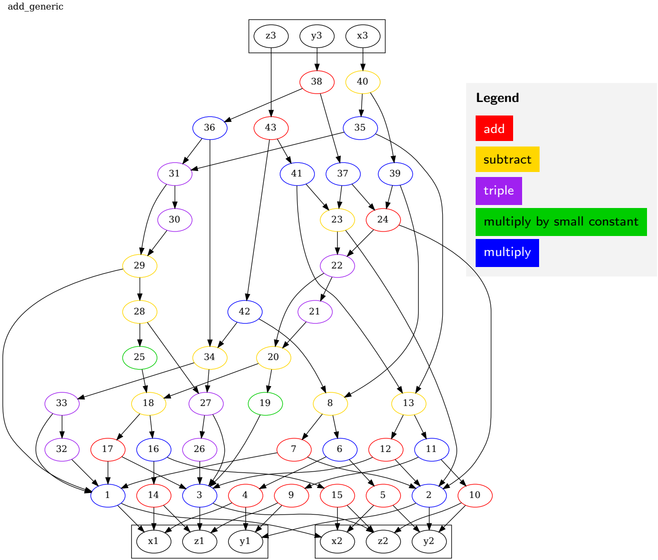
dbl_4x (3M + 4c)



Legend

- add
- subtract
- triple
- multiply by small constant
- multiply
- square

Point addition



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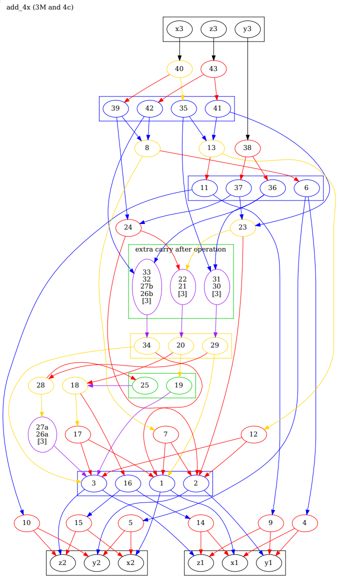

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- Uses $263 \cdot$ **double** $+ 59 \cdot$ **add** operations

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Compared to Curve25519

Table: Cycle counts for Sandy2x and this work.

Implementation	**Sandy Bridge**	**Ivy Bridge**	**Haswell**
Curve25519 (Sandy2x)	159kcc	157kcc	–
this work	390kcc	383kcc	340kcc

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Conclusion: about **2.5**$\times$ slower

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Stuff I left out:

- Ristretto
- Politics
- *Many* implementation details

Thank you! II

The code is at `https://github.com/dsprenkels/curve13318`

Extra reading:

- My thesis: `https://dsprenkels.com/files/thesis-20190311.pdf`
- Monero vulnerability (1): `https://nickler.ninja/blog/2017/05/23/exploiting-low-order-generators-in-one-time-ring-signatures/`
- Monero vulnerability (2): `https://moderncrypto.org/mail-archive/curves/2017/000898.html`

Find me through:

- Email: amber@electricdusk.com
- PGP key: `951D 6F6E C19E 5D87 1A61  A7F4 1445 C075 FFD5 68CD`

References I

- Barreto, P.S.L.M.: on Twitter (May 2017), https://twitter.com/pbarreto/status/869103226276134912
- Bernstein, D.J.: Curve25519: New Diffie-Hellman speed records. pp. 207–228 (2006), https://cr.yp.to/ecdh/curve25519-20060209.pdf
- Bernstein, D.J., Duif, N., Lange, T., Schwabe, P., Yang, B.Y.: High-speed high-security signatures. pp. 124–142 (2011), https://ed25519.cr.yp.to/ed25519-20110926.pdf
- Chou, T.: Sandy2x: New Curve25519 speed records. pp. 145–160 (2016), https://www.win.tue.nl/~tchou/papers/sandy2x.pdf
- Genkin, D., Valenta, L., Yarom, Y.: May the Fourth Be With You: A microarchitectural side channel attack on several real-world applications of Curve25519. pp. 845–858 (2017), https://eprint.iacr.org/2017/806.pdf
- Karatsuba, A., Ofman, Y.: Multiplication of many-digital numbers by automatic computers. Dokl. Akad. Nauk SSSR 145(2), 293–294 (1962), http://www.mathnet.ru/php/getFT.phtml?jrnid=dan&paperid=26729&what=fullt&option_lang=eng

References II

Kaufmann, T., Pelletier, H., Vaudenay, S., Villegas, K.: When constant-time source yields variable-time binary: Exploiting Curve25519-donna built with MSVC 2015. pp. 573–582 (2016), https://infoscience.epfl.ch/record/223794/files/32_1.pdf

Koblitz, N.: Elliptic curve cryptosystems. Math. Comp. 48, 209–209 (1987), https://www.ams.org/journals/mcom/1987-48-177/S0025-5718-1987-0866109-5/S0025-5718-1987-0866109-5.pdf

luigi1111, "fluffypony" Spagni, R.: Disclosure of a major bug in cryptonote based currencies (May 2017), https://src.getmonero.org/2017/05/17/disclosure-of-a-major-bug-in-cryptonote-based-currencies.html

Miller, V.S.: Use of elliptic curves in cryptography. pp. 417–426 (1986), https://www.researchgate.net/profile/Victor_Miller/publication/227128293_Use_of_Elliptic_Curves_in_Cryptography/links/0c96052e065c94b47c000000/Use-of-Elliptic-Curves-in-Cryptography.pdf

Perrin, T.: Subject: [curves] CryptoNote and equivalent points (May 2017), https://moderncrypto.org/mail-archive/curves/2017/000898.html

Renes, J., Costello, C., Batina, L.: Complete addition formulas for prime order elliptic curves. pp. 403–428 (2016), http://eprint.iacr.org/2015/1060

References III

Schnorr, C.P.: Efficient signature generation by smart cards 4(3), 161–174 (Jan 1991), https://www.researchgate.net/profile/Claus_Schnorr/publication/227088517_Efficient_signature_generation_by_smart_cards/links/0046353849579ce09c000000/Efficient-signature-generation-by-smart-cards.pdf

Double-and-add algorithm

```
function DoubleAndAdd(k, P)                          ▷ Compute [k]P
    R ← 𝒪
    for i from n − 1 down to 0 do
        R ← [2]R                                     ▷ Doubling
        if k_i = 1 then
            R ← R + P                                ▷ Addition
        else
            R ← R + 𝒪                                ▷ Addition
        end if
    end for
    return R
end function
```

Fixed-window double-and-add

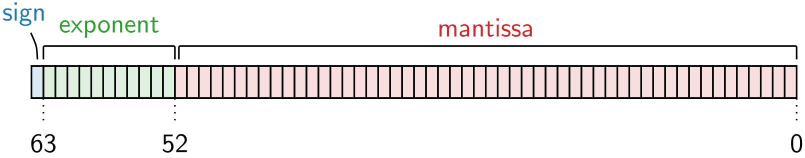
function $\textsc{FixedWindow}(k, P)$ ▷ Compute $[k]P$
  $k' \leftarrow \textsc{Windows}_w(k)$
  Precompute $([2]P, \ldots, [2^w - 1]P)$
  $R \leftarrow \mathcal{O}$
  for i **from** $\frac{n}{w} - 1$ **down to** 0 **do**
    for j **from** 0 **to** $w - 1$ **do**
      $R \leftarrow [2]R$ ▷ w doublings
    end for
    if $k'_i \neq 0$ **then**
      $R \leftarrow R + [k'_i]P$ ▷ Addition
    else
      $R \leftarrow R + \mathcal{O}$ ▷ Addition
    end if
  end for
  return R
end function

Signed double-and-add

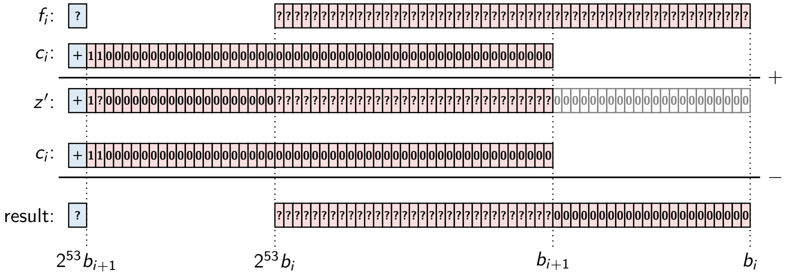
```
function SignedFixedWindow(k, P)                        ▷ Compute [k]P
    k' ← RecodeSigned(Windows_w(k))
    Precompute ([2]P, ..., [2^(w−1)]P)
    R ← O
    for i from n/w − 1 down to 0 do
        for j from 0 to w − 1 do
            R ← [2]R                                    ▷ w doublings
        end for
        if k'_i > 0 then
            R ← R + [k'_i]P                             ▷ Addition
        else if k'_i < 0 then
            R ← R − [−k'_i]P                            ▷ Addition
        else
            R ← R + O                                   ▷ Addition
        end if
    end for
    return R
end function
```

Implemented signed double-and-add

function SCALARMULTIPLICATION(k, P) ▷ Compute $[k]P$
  $\mathbf{T} \leftarrow (\mathcal{O}, P, \dots, [16]P)$ ▷ Precompute $([2]P, \dots, [16]P)$
  $k' \leftarrow$ RECODESIGNED(WINDOWS$_5$(k))
  $R \leftarrow \mathcal{O}$
  for i **from** 50 **down to** 0 **do**
    for j **from** 0 **to** 4 **do**
      $R \leftarrow [2]R$ ▷ 5 doublings
    end for
    if $k'_i < 0$ **then**
      $R \leftarrow R - \mathbf{T}_{-k'_i}$ ▷ Addition
    else
      $R \leftarrow R + \mathbf{T}_{k'_i}$ ▷ Addition
    end if
  end for
  return R ▷ $R = (X_R : Y_R : Z_R)$
end function


sign
exponent
mantissa
63
52
0

Depiction of TOP(f)



Signed windows

$$k = \underbrace{1011}_{k'_3}\underbrace{0010}_{k'_2}\underbrace{0110}_{k'_1}\underbrace{1110}_{k'_0}$$

Signed window recoding

$$k = 1011 \quad 0010 \quad 0110 \quad 1110$$

$$\downarrow$$

k''_4	k''_3	k''_2	k''_1	k''_0
1	−101	010	111	−010